

# Quantum phase transition in many-flavor supersymmetric QED<sub>3</sub>

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We study  $\mathcal{N} = 4$  supersymmetric QED in three dimensions, on a three-sphere, with  $2N$  massive hypermultiplets and a Fayet-Iliopoulos parameter. We identify the exact partition function of the theory with a conical (Mehler) function. This implies a number of analytical formulas, including a recurrence relation and a second-order differential equation, associated with an integrable system. In the large  $N$  limit, the theory undergoes a second-order phase transition on a critical line in the parameter space. We discuss the critical behavior and compute the two-point correlation function of a gauge invariant mass operator, which is shown to diverge as one approaches criticality from the subcritical phase. Finally, we comment on the asymptotic  $1/N$  expansion and on mirror symmetry.

The study of quantum electrodynamics in three dimensions (QED<sub>3</sub>) has been a subject of interest since the early 1980s, due to its connection to finite-temperature QCD via dimensional reduction and to the fact that, as QCD in four dimensions, QED<sub>3</sub> also exhibits spontaneous chiral symmetry breaking and confinement [1].

The theory has experienced a remarkably renewed interest in recent years due to, in great part, the relevance of relativistic field theories of particles moving in two dimensions in the description of the pseudogap phase of cuprates [2], the spin-liquid phase of quantum antiferromagnets [3] and the low-energy electronic excitations of graphene [4]. This reinvigorated relevance of the dynamics of a  $U(1)$  gauge field and  $N_f$  fermions in three dimensions extends to the case with supersymmetry, where the theory appears for example in descriptions of the physics of half-filled Landau levels in terms of Dirac fermions [5], in 3d Bosonization [6] and non-perturbative descriptions of renormalization group flows [7]. Supersymmetric  $U(1)$  theories in three dimensions can be related to the study of quantum phase transitions in quantum antiferromagnets and provide examples of quantum phase transitions beyond the Landau-Ginzburg paradigm [8]. In these discussions, the existence of mirror symmetry in the supersymmetric gauge theory is important [6, 8] and it is precisely the recent, more detailed, analysis of dualities that has bolstered great interest in QED<sub>3</sub>. In particular for example, it has been recently shown that the fermionic vortex of QED<sub>3</sub> is a free Dirac fermion [9, 10]. Around this result they lurk a number of connections between topological insulators, spin liquids and quantum Hall physics, making QED<sub>3</sub> a subject of considerable physical interest.

At the same time, the development of tools to studying supersymmetric gauge theories on curved manifolds, in particular localization [11, 12], have increased the means at our disposal to obtain exact analytical results. Examples of works that use localization and the F-theorem

in the study of QED<sub>3</sub> (with or without supersymmetry), includes [13–15]. In this paper, we shall study supersymmetric  $U(1)$  gauge theory on  $S^3$  but, instead of massless matter as in [14], or the cases studied in [7, 15], we include massive  $\mathcal{N} = 4$  hypermultiplets and a Fayet-Iliopoulos (FI) term. As we will see, this leads to a dramatic change in the dynamics of the theory. For the sake of simplicity, we consider the case of  $N$  hypermultiplets with mass  $m$  and  $N$  hypermultiplets with mass  $-m$ . More general mass configurations will be discussed at the end. The total number of flavors  $N_f$  is therefore  $2N$ . In addition, as mentioned, there is a FI term and it is actually the interplay between the difference of masses and the FI parameter  $\eta$  the one which is responsible for producing novel behavior. In particular, it is responsible for the emergence of a second-order quantum phase transition in the large  $N$  limit.

We thus consider an  $\mathcal{N} = 4$  supersymmetric  $U(1)$  theory consisting of  $2N$  massive  $\mathcal{N} = 4$  (flavor) hypermultiplets ( $N$  of mass  $m$  and  $N$  of mass  $-m$ ), coupled to an  $\mathcal{N} = 4$  vector multiplet. Localization readily leads to an integral representation for the partition function [16]

$$\begin{aligned} Z_{\text{QED}_3} &= \int_{-\infty}^{\infty} dx \frac{e^{i\eta x}}{[2 \cosh(\frac{x+m}{2}) 2 \cosh(\frac{x-m}{2})]^N} \\ &= 2^{-N} \int_{-\infty}^{\infty} dx \frac{e^{i\eta x}}{[\cosh x + \cosh m]^N}. \end{aligned} \quad (1)$$

In what follows we drop the  $2^N$  factor, which is inessential to our discussion and we have set the radius of  $S^3$  to  $r = 1/(2\pi)$ . In the case when the hypermultiplets have masses  $m_1$  and  $m_2$ , the parameter  $m$  is  $m = (m_1 - m_2)/2$ . Thus, in the discussion below, increasing  $m$  corresponds to separating the two mass scales.

Notice that, by writing the integral representation (1) in the latter form, one can immediately identify it as the integral representation of a conical (Mehler) func-

tion [17], which is an associated Legendre function with a complex index. We find

$$Z_{\text{QED}_3} = \sqrt{2\pi} \frac{\Gamma(N+i\eta)\Gamma(N-i\eta)}{\Gamma(N)(\sinh(m))^{N-\frac{1}{2}}} P_{-\frac{1}{2}+i\eta}^{\frac{1}{2}-N}(\cosh(m)) . \quad (2)$$

It can also be conveniently represented in terms of an hypergeometric function:

$$Z_{\text{QED}_3} = \frac{\sqrt{2\pi} \Gamma(N+i\eta)\Gamma(N-i\eta)}{\Gamma(N)\Gamma(N+\frac{1}{2})(1+z)^{N-\frac{1}{2}}} \times {}_2F_1\left(\frac{1}{2}-i\eta, \frac{1}{2}+i\eta, N+\frac{1}{2}; \frac{1}{2}(1-z)\right). \quad (3)$$

with  $z \equiv \cosh(m)$ . In specific cases, the expression simplifies. In particular for two and four flavors, we find

$$Z_{\text{QED}_3}^{N=1} = \frac{2\pi \sin(m\eta)}{\sinh(m) \sinh(\pi\eta)}, \quad (4)$$

$$Z_{\text{QED}_3}^{N=2} = \frac{2\pi (\cosh m \sin(m\eta) - \eta \sinh m \cos(m\eta))}{\sinh^3(m) \sinh(\pi\eta)}. \quad (5)$$

These can also be obtained from residue integration [18]. The expression (4) already exhibits some of the general properties of the conical function and, hence, of the partition function, such as the oscillatory behavior, which depends on both  $m$  and  $\eta$ . It is well known that supersymmetric QED<sub>3</sub> with 2 flavors is self-dual [19]. Notice that indeed we find that (4) is invariant under the exchange  $m \leftrightarrow \pi\eta$  implied by the duality transformation.

The oscillatory behavior is related to the fact that the function has an infinite number of zeros, all of them real, precisely in the physical region  $m \geq 0$  [17] and the function is monotonic until the appearance of the first zero of the function, after which the behavior is oscillatory. This transition between a monotonic and an oscillatory region when the first zero appears, in the large  $N$  limit, becomes a phase transition, which we characterize below by computing the saddle points of (1).

The identification with a conical function and the ensuing hypergeometric representation has interesting consequences. To begin with, from a standard recurrence relation for the Legendre functions, we obtain that the partition function also satisfies a recurrence relation:

$$(2N-1) \cosh(m) Z_N = \frac{(N-1)^2 + \eta^2}{N-1} Z_{N-1} + N \sinh^2(m) Z_{N+1} \quad (6)$$

For short, here we defined  $Z_N \equiv Z_{\text{QED}_3}(m, \eta, N)$ . By this formula, we can easily generate any  $Z_N$  from the above expressions (4), (5) for  $N=1$  and  $N=2$ . In addition, the representation (3) can be used to study a small mass expansion, since the radius of convergence of a Gauss hypergeometric function is  $|x| < 1$  in the variable. In the massless limit, the hypergeometric becomes 1 and  $Z_{\text{QED}_3}$  is given by the first line in (3). In order to find

the large mass behavior, we use an Euler hypergeometric transformation and write the partition in the form

$$Z_{\text{QED}_3} = \frac{\Gamma(-i\eta)\Gamma(N+i\eta)}{2^{i\eta}\Gamma(N)(\cosh(m)+1)^{N+i\eta}} \times {}_2F_1\left(\frac{1}{2}+i\eta, N+i\eta, 1+2i\eta; \text{sech}^2\frac{m}{2}\right) + c.c. \quad (7)$$

Using (7) we then obtain

$$Z_{\text{QED}_3}^{m \gg 1} = \frac{2^N \Gamma(-i\eta)\Gamma(N+i\eta)}{\Gamma(N)} e^{-m(N+i\eta)} + c.c.. \quad (8)$$

Another non-trivial consequence of the relation of the partition function to a hypergeometric function is the fact that then  $Z_N$  satisfies a second-order differential equation

$$\frac{d^2 Z_N}{dm^2} + 2N \coth(m) \frac{dZ_N}{dm} + (\eta^2 + N^2) Z_N = 0. \quad (9)$$

By defining  $\tilde{Z}_{\text{QED}_3} = (\sinh(m))^N Z_{\text{QED}_3}$ , this equation can be written as a Schrödinger equation with an hyperbolic Pöschl-Teller potential [20], which is a well-known solvable 1d quantum mechanical problem

$$\frac{d^2 \tilde{Z}_{\text{QED}_3}}{dm^2} + \left(\eta^2 + \frac{N(1-N)}{\sinh^2 m}\right) \tilde{Z}_{\text{QED}_3} = 0.$$

The present theory has a large  $N$  limit, with fixed  $\lambda \equiv \eta/N$ . In this limit, the partition function (1) can be computed by the saddle-point method. The integrand in (1) can be written as  $e^{-NS(\lambda)}$  where the *action*  $S$  is

$$S(\lambda, x, z) = -i\lambda x + \log(\cosh x + \cosh m).$$

The saddle-point equation is then

$$-i\lambda + \frac{\sinh x}{\cosh x + \cosh m} = 0,$$

which has as solutions,

$$x_{1,2} = \log\left(\frac{-\lambda \cosh m \pm i\Delta}{i + \lambda}\right) + 2\pi i n, \quad (10)$$

where  $n \in \mathbb{Z}$  and  $\Delta \equiv \sqrt{1 - \lambda^2 \sinh^2 m}$ . In what follows we show that the theory undergoes a large  $N$  phase transition at  $\lambda_c \equiv 1/\sinh m$ , or, more generally, at the critical line  $\lambda \sinh(m) = 1$  in the  $(\lambda, m)$  space, where  $\Delta = 0$ . *Subcritical phase* ( $\lambda \sinh(m) < 1$ ). In this case all saddle points lie on the imaginary axis. We find that the saddle point  $x_1$  with  $n = 0$  is the relevant one and, to leading order for large  $N$ , the partition function becomes

$$Z_{\text{QED}_3} \approx \frac{\sqrt{2\pi}}{\sqrt{NS''(x_1)}} \exp(-NS(x_1)). \quad (11)$$

where  $S''(z, x) = (z \cosh(x) + 1) / (\cosh(x) + z)^2$ . We numerically checked that, for large  $N$ , this formula reproduces the analytic expression (3) with great accuracy. In the subcritical phase, the large  $N$  free energy

$F = -\log Z_{\text{QED}_3}$  is given by  $NS(x_1)$ . We obtain

$$F_{\text{sub}} = -\frac{i\lambda N}{2} \log\left(\frac{(-z\lambda + i\Delta)(i - \lambda)}{(z\lambda + i\Delta)(i + \lambda)}\right) + N \log\left(\frac{z + \Delta}{1 + \lambda^2}\right)$$

*Supercritical phase* ( $\lambda \sinh(m) > 1$ ). The two saddle points move to the complex plane, with  $x_2 = -x_1^*$ . The action is complex, with  $\text{Re}(S(x_1)) = \text{Re}(S(x_2))$ ,  $\text{Im}(S(x_1)) = -\text{Im}(S(x_2))$ . Therefore, both saddle points contribute with equal weights and need to be taken into account. The partition function  $Z = Z_{\text{QED}_3}$  is now

$$Z \approx \sqrt{\frac{2\pi}{N}} e^{-N \text{Re}(S(x_1))} \left( \frac{e^{-iN \text{Im}(S(x_1))}}{\sqrt{S''(x_1)}} + c.c. \right).$$

For  $N \gg 1$ , this expression agrees with the exact analytic expression (3). For large mass, it is also in precise agreement with the large mass formula (8). Taking the log to get the free energy, we see that at large  $N$ , the leading contribution proportional to  $N$  is given by the real part of the action  $F \approx N \text{Re}(S(x_1))$ , giving

$$F_{\text{super}} = \frac{N}{4} \left( 2 \log \frac{z^2 - 1}{1 + \lambda^2} - 2i\lambda \log \frac{\lambda - i}{\lambda + i} \right).$$

The free energy and its first derivative are continuous at the critical point, while the second derivative gives

$$\frac{d^2 F}{d\lambda^2} = \frac{N}{1 + \lambda^2} \left( 1 + \frac{\cosh(m)}{\sqrt{1 - \lambda^2 \sinh^2(m)}} \right), \quad \lambda < \lambda_c,$$

$$\frac{d^2 F}{d\lambda^2} = \frac{N}{1 + \lambda^2}, \quad \lambda \geq \lambda_c.$$

Thus  $d^2 F/d\lambda^2$  is discontinuous, implying a second-order phase transition. In addition, in the subcritical regime the susceptibility  $\chi = -\frac{d^2 F}{d\lambda^2}$  diverges as the critical line is approached,  $\chi \sim (\lambda_c - \lambda)^{-\gamma}$ ,  $\gamma = 1/2$ , which is a recurrent behavior in 2nd-order phase transitions. The critical behavior is shown in figs. 1a and 1b for fixed  $m = 1$  and in fig. 2 for the whole phase diagram.

Next, consider the analytic properties of the free energy in crossing the critical line by varying the mass parameter at fixed coupling  $\lambda$ . By differentiating the free energy with respect to the mass  $m$ , one generates correlators of the gauge invariant mass operator [15]

$$J_3 = \frac{1}{N} \left( \tilde{Q}_{1,i} Q_1^i - \tilde{Q}_{2,i} Q_2^i \right),$$

where  $Q_1$  are the hypermultiplets of mass  $m$  and  $Q_2$  the hypermultiplets of mass  $-m$ . Because of supersymmetry, these correlators are independent of the position [15]. For example, for the simple  $N = 1$  case, we have that

$$\begin{aligned} \langle J_3 \rangle &\propto \frac{dF}{dm} = \eta \cot(m\eta) - \coth(m), \\ \langle J_3 J_3 \rangle - \langle J_3 \rangle \langle J_3 \rangle &\propto \frac{d^2 F}{dm^2} = -\frac{\eta^2}{\sin^2(m\eta)} + \frac{1}{\sinh(m\eta)}. \end{aligned}$$

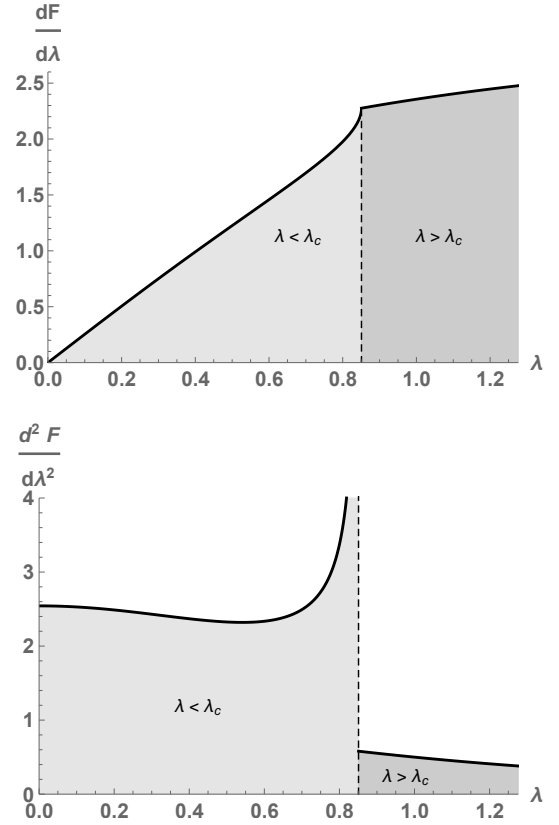


FIG. 1: (a) Behavior of  $dF/d\lambda$ . (b) Discontinuity of  $d^2 F/d\lambda^2$  at the transition point ( $m = 1$ ).

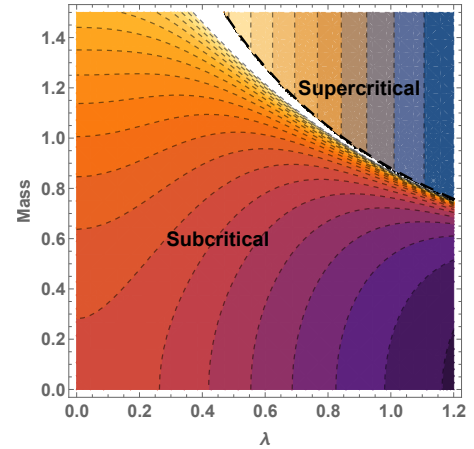


FIG. 2: Phase diagram. The critical line (dashed)  $\lambda \sinh(m) = 1$  separates the two phases. The plot also shows the contour lines of  $d^2 F/d\lambda^2$  (increasing from dark to light).

This extends a result of [15] to the case  $\eta \neq 0$ . Returning to the large  $N$  free energy, we find that  $\langle J_3 \rangle$  is continuous,

whereas

$$\begin{aligned} \left( \frac{d^2 F}{dm^2} \right)_{\lambda < \lambda_c} &= \frac{1}{N \sinh^2 m} \left( 1 - \frac{\cosh m}{\sqrt{1 - \lambda^2 \sinh^2 m}} \right) \\ \left( \frac{d^2 F}{dm^2} \right)_{\lambda > \lambda_c} &= \frac{1}{N \sinh^2 m} . \end{aligned}$$

Thus  $d^2 F/dm^2$  is discontinuous, implying a discontinuity in the two-point function of the operator  $J_3$ . Moreover, the two-point correlation function diverges as the critical line is approached from the subcritical phase.

The theory has an asymptotic  $1/N$  expansion, which we now briefly outline. For concreteness, we consider the subcritical phase. The first  $1/N$  correction arises from the term  $(x - x_1)^4$  in the expansion of the integrand of (1) around  $x_1$ . A closely related expansion of the conical functions in inverse powers of  $(N - 1/2)$  was discussed in [17, 21]. An interesting approach is described in [7].

An elegant treatment, which exhibits the asymptotic character of the  $1/N$  series, is as follows. We introduce a new integration variable by the transformation

$$S(x) - S_0 = \beta t . \quad (12)$$

This leads to

$$Z_{\text{QED}_3} = \beta e^{-NS_0} \int_{\mathcal{C}} dt e^{-N\beta t} \mathcal{B}(t), \quad \mathcal{B}(t) \equiv \frac{1}{S'(x(t))}, \quad (13)$$

with  $S_0 \equiv S(x_1)$ ,  $\beta \equiv S''(x_1)/2$ . The contour  $\mathcal{C}$  in the complex  $t$ -plane is determined by the transformation (12) in varying  $x$  from  $-\infty$  to  $\infty$  (see fig. 3). The contour surrounds singularities at  $t_1^{(n)}$  and  $t_2^{(n)}$  lying on the positive real axis, which are associated with the saddles at  $x_1$ , with  $n = 0, 1, 2, \dots$ ,  $x_2$ , with  $n = 1, 2, \dots$ . All singularities in  $\mathcal{B}(t)$  are branch points of the form  $(t - t_{1,2}^{(n)})^{-1/2}$ . The  $1/N$  expansion is generated upon Taylor expanding  $\mathcal{B}(t)$  in powers of  $t$ ,

$$\mathcal{B}(t) = \frac{1}{2\beta\sqrt{t}} \sum_{k=0}^{\infty} b_k t^k, \quad b_0 = 1 .$$

This expansion has a finite radius of convergence, determined by the location of the singularity that is closest to the origin. We are left with the integral

$$\int_{\mathcal{C}} dt e^{-N\beta t} t^{k-1/2} = 2 \int_0^{\infty} dt e^{-N\beta t} t^{k-1/2} = \frac{2\Gamma(k+1/2)}{(\beta N)^{k+1/2}},$$

where we have deformed the contour to the positive real axis (note that in this integral there is no singularity on the positive real axis). Thus we get the asymptotic series

$$Z_{\text{QED}_3} = \frac{e^{-NS_0}}{(\beta N)^{1/2}} \sum_{k=0}^{\infty} b_k \frac{\Gamma(k+1/2)}{\beta^k N^k} . \quad (14)$$

The term  $k = 0$  just reproduces the earlier formula (11).

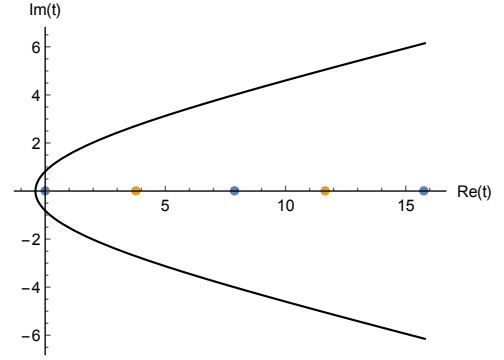


FIG. 3: Integration contour  $\mathcal{C}$  in (13) and location of singularities of the integrand ( $m = 1$ ,  $\lambda = 0.3\lambda_c$ ).

Here we have expanded  $\mathcal{B}(t)$  around  $t = 0$ . By expanding  $\mathcal{B}(t)$  around some  $t_{1,2}^{(n)}$ ,  $n = 1, 2, \dots$  one finds an extra factor  $e^{-2\pi n\lambda}$  coming from  $e^{-NS(t)}$ . The presence of an infinite number of saddle-points suggests that the  $1/N$  expansion can be more conveniently treated in terms of resurgent trans-series. In deforming the contour, one crosses Stokes discontinuities which may imply resurgent relations in the different trans-series coefficients (see [22–26] for examples). It would be extremely interesting to understand the origin of non-perturbative effects that render the large  $N$  expansion asymptotic, as well as the resurgent properties of the series and how the existence of a phase transition is encoded in the  $1/N$  expansions below and above the phase transition.

Now consider more general masses. The theory with  $N$  hypermultiplets of mass  $m_1$  and  $N$  hypermultiplets of mass  $m_2$  is equivalent to the one we discussed. By a shift in the integration variable, one gets the same partition function (1) with an extra phase  $e^{-i\eta m_+}$  and  $m$  replaced by  $m_-$ , with  $m_{\pm} = (m_1 \pm m_2)/2$ . In general, for  $N_f$  flavors, there are  $N_f - 1$  mass parameters associated with the Cartan generators of  $SU(N_f)$  flavor symmetry, satisfying  $\sum_i m_i = 0$ . One can have independent parameters  $m_1$  and  $m_2$  by adding an extra hypermultiplet of mass  $m_3 = -N(m_1 + m_2)$ . At large  $N$ , this decouples (its 1-loop partition function becomes a constant,  $1/\cosh m_3$ ) and the large  $N$  physics is then the same as in (1). More general mass assignments with similar phase transition are possible. The reason is that the mechanism that triggers the phase transition is also at work in more general cases: on the imaginary axis, the 1-loop partition function provides a periodic potential with infinite number of vacua; as the constant force, represented by the FI parameter, is increased, there is a critical point where this overcomes the maximum force from the periodic potential. Beyond this point, equilibrium is not possible and the saddle points move to the complex plane.

For 3d  $\mathcal{N} = 4$  theories, mirror symmetry involves two or more theories with a different UV description flowing

to the same superconformal point in the IR. Mirror symmetry interchanges Coulomb and Higgs branches of the theory, where FI parameters are interchanged with some linear combination of mass parameters [19]. The present theory is known to be dual to a  $A_{N-1}$  quiver gauge theory [19, 27]. Particularizing to our model, we see that the dual theory is a  $U(1)^{2N-1}$  quiver gauge theory with a FI parameter  $2m$  and a single mass for all hypermultiplets  $-\eta/2N$ . Our results show that the quiver gauge theory also has a novel type of phase transition in the limit when the number of *quiver nodes* goes to infinity.

To conclude, the partition function of supersymmetric QED<sub>3</sub> with FI term is given in terms of the conical function (3). It is remarkable that this simple formula encapsulates very rich physical phenomena such as large  $N$  phase transitions, asymptotic  $1/N$  expansion, emergence of complex saddle points, non-perturbative effects and aspects of mirror symmetry.

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